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1990 J. Phys. A: Math. Gen. 23 3809

(http://iopscience.iop.org/0305-4470/23/16/026)

J. Phys. A: Math. Gen. 23 (1990) 3809-3823. Printed in the UK

Stationary distributions for systems with competing creation–annihilation dynamics*

A I López-Lacomba, P L Garrido† and J Marro

Departamento de Física Aplicada, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain

Received 8 February 1990

Abstract. We have developed a simple, systematic method to investigate the existence of stationary probability distributions (SPDs) for interacting particle (or spin) lattice systems exhibiting steady non-equilibrium states. The latter originate in a competition between several creation-annihilation (spin-flip) kinetic mechanisms, say each acting presuming a different bath temperature, particle (spin) interactions strength or chemical potential (magnetic field). It follows the existence of SPDs for a large class of these systems which may thus be studied by simply applying the techniques of equilibrium theory. The method is illustrated with several examples bearing a practical interest.

1. Introduction and definition of model

Consider for simplicity a regular lattice $\Omega=\{\boldsymbol{r}_i;\ i=1,2,\ldots,N\}$ in a d-dimensional space with occupation or spin variables $s_i\equiv s_{\boldsymbol{r}i}=\pm1$ at each lattice site. Denote by $\boldsymbol{s}\equiv\{s_{\boldsymbol{r}};\ \boldsymbol{r}\in\Omega\}$ any configuration, by $\boldsymbol{S}\equiv\{\boldsymbol{s}\}$ the set of (2^N) possible configurations, by $P(\boldsymbol{s};\ t)$ the probability of $\boldsymbol{s}$ at time t, and by $P(\boldsymbol{s}|\boldsymbol{s}';\ t)$ the probability of a transition from configuration $\boldsymbol{s}'$ to configuration $\boldsymbol{s}$ in time interval t. The system evolves in time as implied by the Kolmogorov equations (Haken 1977, van Kampen 1981, Ligget 1985):

$$\mathrm{d}P(\boldsymbol{s}|\boldsymbol{s}';\ t)/\mathrm{d}t=\sum_{\boldsymbol{s}''\in\boldsymbol{s}}c(\boldsymbol{s}|\boldsymbol{s}'')P(\boldsymbol{s}''|\boldsymbol{s}';\ t) \tag{1.1a}$$

$$\mathrm{d}P(\boldsymbol{s}|\boldsymbol{s}';\ t)/\mathrm{d}t=\sum_{\boldsymbol{s}''\in\boldsymbol{S}}P(\boldsymbol{s}|\boldsymbol{s}'';\ t)c(\boldsymbol{s}''|\boldsymbol{s}'). \tag{1.1b}$$

where $c(\boldsymbol{s}|\boldsymbol{s}')$ are the elements of a matrix $\boldsymbol{c}$ of transition rates per unit time from $\boldsymbol{s}'$ to $\boldsymbol{s}$ satisfying

$$c(\boldsymbol{s}|\boldsymbol{s}')\geqslant0\qquad\text{for all } \boldsymbol{s}\neq\boldsymbol{s}' \tag{1.2a}$$

and

$$\sum_{\boldsymbol{s}\in\boldsymbol{S}}c(\boldsymbol{s}|\boldsymbol{s}')=0. \tag{1.2b}$$

* Partially supported by Dirección General de Investigación Científica y Técnica, Project PB88-0487, Plan Andaluz de Investigación (Junta de Andalucía), and Commission of the European Communities, Contract CI1.0409.

† Present address: Department of Mathematics, Hill Center, Rutgers University, New Brunswick, NJ 08903, USA.

0305-4470/90/163809+15$03.50

Under those conditions, equations (1.1) admit a unique solution with the following properties:

$$P(s|s';\, t) \geqslant 0 \tag{1.3a}$$

$$\sum_{s \in S} P(s|s';\, t) = 1 \tag{1.3b}$$

$$P(s|s';\, t) = \sum_{s'' \in S} P(s|s'';\, t-\tau) P(s''|s';\, \tau) \tag{1.3c}$$

and

$$\lim_{t \to 0^+} P(s|s';\, t) = \delta_{s,s'} \tag{1.3d}$$

for all $\tau \in (0, t)$, $s, s' \in S$ and $t \geqslant 0$. Properties (1.3a) and (1.3b) characterise the solutions of (1.1) as transition probabilities, (1.3c) is the Chapman–Kolmogorov or Smoluchowski equation describing a *homogeneous Markov process*, and (1.3d) characterises $c(s\,|\,s')$ as a true transition rate; the latter fact follows from (1.1a) and (1.3d) in the limit $t \to 0^+$. It then results that the probability distribution $P(s;\, t)$ satisfies:

$$\mathrm{d}P(s;\, t)/\mathrm{d}t = \sum_{s' \in S} c(s|s') P(s';\, t). \tag{1.4}$$

This is the so-called master equation which is the most familiar prescription for a homogeneous Markov process.

We are particularly interested here in *local* kinetic processes such that c involves a *series* of competing creation–annihilation or spin-flip mechanisms each producing, as in the so-called Glauber (1963) dynamics, the change $s_i \to s_i$ of the variable at a site r_i thus generating a new configuration s' (to be denoted specifically either as s^{ri} or else as s^i) from s with a given probability per unit time, $c(s^r|s)$. That is, we shall assume in the following that

$$c(s'|s) = -m^{-1} \sum_{k=1}^{m} \sum_{r} \prod_{x \neq r} \delta_{s_x, s'_x} c_{\varphi_k}(s^r|s) s_r s'_r \tag{1.5}$$

where $r \in \Omega$, and φ_k represents the value of a given parameter such as temperature, chemical potential, magnetic field, sign or strength of interactions, or any combination of them, etc. For $m = 1$ and certain choices c (see section 2), the system will evolve towards the equilibrium state, while a dynamics with $m \geqslant 2$ will in general induce stationary non-equilibrium states, as occurs when a system is not isolated but acted on by some external agent.

Our main objective in this paper is to describe and apply a simple and systematic method to find stationary solutions, $P^{\mathrm{st}}(s)$, of (1.1) or (1.4) for a system evolving via the competing kinetic process c defined by (1.2) and (1.5). We shall also extract other relevant general information concerning such complex dynamical systems, and we shall illustrate the method in the case of some physically interesting, non-trivial non-equilibrium models. It thus follows the existence of a class of such systems, including at least the ones for which our formalism yields an explicit stationary probability distribution $P^{\mathrm{st}}(s)$, which may be studied by simply applying the standard, powerful methods of equilibrium theory. A preliminary attempt to deal with similar general questions was presented before (Garrido and Marro 1989); we pay now more attention to some formal problems and generalise somewhat our results and original procedure so that it can readily be applied to a wider range of systems. In particular, the method here may be used successfully in the analysis of non-equilibrium stationary states and phase transitions in many-lattice systems, and it may also be of interest in the study of some 'cellular automata' (Lebowitz *et al* 1990). Further applications of the method, its generalisation to allow as well for diffusion (Garrido and Marro 1989), and the detailed study of certain specific systems with great practical interest, such as solvable kinetically disordered systems having some relevance in relation with spin-glass, random-field or magnetically diluted models, will be reported elsewhere. Some related efforts may be found in the recent literature (e.g. Künsch 1984, Grinstein *et al* 1985, Wang and Lebowitz 1988, Browne and Kleban 1989, Droz *et al* 1989).

2. Further definitions and space

The asymptotic behaviour of $P(s; t)$ as implied by (1.4) and a given kinetic process (1.5), may be very varied in principle. For instance, the limit of $P(s; t)$ as $t \to \infty$ may not exist and, when it exists, it may or may not depend on the initial probability distribution $P(s; 0)$. This is intimately related to the existence and number of stationary solutions

$$c \cdot P^{\mathrm{st}} = 0 \tag{2.1}$$

where $P^{\mathrm{st}} = \{P^{\mathrm{st}}(s)\}_{s \in S}$ is some probability distribution. As a matter of fact, when at least one solution of (2.1) exists, $P(s; t)$ will have a limit as $t \to \infty$ for at least one initial distribution, $P(s; 0) \equiv P^{\mathrm{st}}(s)$. That problem may be analysed by noticing that, when P^{st} exists, one may define an object $E(s)$ such that

$$P^{\mathrm{st}}(s) = Z^{-1} \exp\{-E(s)\} \qquad Z \equiv \sum_{s \in S} \exp\{-E(s)\}. \tag{2.2}$$

We are naturally assuming that $P^{\mathrm{st}}(s) > 0$ for all s. It thus follows that $E(s)$ is analytic and one may write quite generally that

$$E(s) = \sum_{k=1}^{N} \sum_{(i_1 \ldots i_k)}' J^{(k)}_{i_1 \ldots i_k} s_{i_1} \ldots s_{i_k} \tag{2.3}$$

where Σ' sums over every set of k lattice sites in the system. The object $E(s)$, however, may be rather useless unless it has some appropriate short-range nature, e.g. it may occur in general that $E(s)$ involves infinite coefficients $J^{(k)}$ for the most relevant case of a macroscopic ($N \to \infty$) system. Consequently, we shall be interested in the following on objects $E(s)$ such that

$$J^{(k)}_{i_1 \ldots i_k} = 0 \qquad \text{for all } k \geq k_0 \tag{2.4}$$

where k_0 is independent of N, at least for $N > N_0$. When it exists a unique stationary distribution function P^{st} such that when (2.2)-(2.4) hold, the resulting object $E(s)$ will be termed the *effective Hamiltonian* (EH) of the system.

A familiar realisation of a homogeneous Markov process in a lattice system occurs in the kinetic Ising or Glauber (1963) model for which $m = 1$. This consists of any lattice Ω whose configurations, assuming for the moment that there is no external magnetic field or chemical potential, have a potential or configurational energy given by

$$H(s) = -J \sum_{nn} s_{r_i} s_{r_j} \tag{2.5}$$

where the sum is over nearest-neighbour (NN) pairs of sites. Moreover, the lattice is in contact with a thermal bath at temperature T which induces changes in s such that the system will asymptotically reach the canonical equilibrium state $P^{\mathrm{st}}(s) = \text{constant} \exp\{-H(s)/k_{\mathrm{B}}T\}$. Namely, the thermal bath provokes spin-flips with a prescribed rate depending on the change of the energy (2.5) which would cause the flip:

$$c(s^r|s) = f_r(s) \exp\left[-Ks_r \sum_q s_{r'}\right] = f_r(s) \exp[-\tfrac{1}{2}\delta H] \tag{2.6}$$

where $\delta H \equiv [H(s^r) - H(s)]/k_{\mathrm{B}}T$, $K \equiv J/k_{\mathrm{B}}T$, the sum is over the q nearest neighbours of site r and $f_r(s) = f_r(s^r)(>0)$ as required by the detailed balance condition:

$$c(s^r|s)/c(s|s^r) = \exp(-\delta H) \tag{2.7}$$

which assures approach to equilibrium. For instance, one may take:

$$f_r(s) = \text{constant}\left[\prod_{i=1}^{d} \cosh(2K\sigma^r_{1i})\right]^{-1} \tag{2.8a}$$

where $\sigma^r_{1i} \equiv \frac{1}{2}(s_{r+i} + s_{r-i})$ and i represent unity vectors along each lattice principal direction, which corresponds to the familiar rate first introduced by Glauber (1963) for $d = 1$. Indeed, the original Glauber rate may be generalised to arbitrary dimension as

$$c(s^r|s) = \text{constant}[A_+(s) + s_r A_-(s)] \tag{2.8a'}$$

where

$$A_\pm(s) = \frac{1}{2}\left[\prod_{i=1}^{d} (1 - \alpha\sigma^r_{1i}) \pm \prod_{i=1}^{d} (1 + \alpha\sigma^r_{1i})\right] \qquad \alpha \equiv \tanh(2K). \tag{2.8a''}$$

Also interesting are the following choices:

$$f_r(s) = \text{constant}[\cosh(K)]^{-2d} \tag{2.8b}$$

that is

$$c(s^r|s) = \left[\prod_{i=1}^{d} (1 + \alpha^2\sigma^r_{2i})\right][A_+(s) + s_r A_-(s)]^{-1} \tag{2.8b'}$$

where $\alpha \equiv \tanh(K)$ and $\sigma^r_{2i} \equiv s_{r+i}s_{r-i}$,

$$f_r(s) = \text{constant}[2\cosh(\tfrac{1}{2}\delta H)]^{-1} \tag{2.8c}$$

$$f_r(s) = \exp\left\{-\left|2K \sum_{i=1}^{d} \sigma^r_{1i}\right|\right\} \tag{2.8d}$$

and

$$f_r(s) = \text{constant} \tag{2.8e}$$

which have been used before respectively by de Masi *et al* (1985), Kawasaki (1972), Metropolis *et al* (1953) and by van Beijeren and Schulman (1984) to deal with different problems. Every choice (2.8) satisfies (2.7) and, consequently, each drives the system towards the same stationary state, the equilibrium one. It simply follows in those (trivial) cases that $E(s) = H(s)/k_{\mathrm{B}}T$, and the system properties may be obtained in principle, and sometimes also in practice, from the computation of Z as defined by (2.2).

The situation is, however, more complex and also more interesting and challenging nowadays (see, for instance, Lebowitz *et al* 1988) for a dynamics such as (1.5) with $m \geqslant 2$. That is, the existence of P^{st}, and that of an effective Hamiltonian, is then an open question in general, as stated above. Consequently, specific methods of solution, also having in general an approximate nature, needed to be developed in the near past for each situation. A simple non-trivial example of that, which displays stationary non-equilibrium states characterised by a small set of macroscopic parameters, occurs when one considers (Garrido *et al* 1987):

$$c(\boldsymbol{s}^r|\boldsymbol{s}) = pc_1(\boldsymbol{s}^r|\boldsymbol{s}) + (1-p)c_2(\boldsymbol{s}^r|\boldsymbol{s}) \qquad 0 \leqslant p \leqslant 1 \tag{2.9}$$

where c_i, $i = 1, 2$, are both given by (2.6) but correspond to different temperatures, say T_i. This may be interpreted, for instance, by assuming that the flip of the spin at $\boldsymbol{r}$ is attempted with probability p as if it were in contact with a thermal bath at temperature $T_1 = T - \delta T$ and with probability $1-p$ as if the temperature of the bath including the transition were $T_2 = T + \delta T$ with $T \geqslant \delta T > 0$. In the limit $\delta T \to 0$, one recovers the equilibrium case described above, while $\delta T \neq 0$ may produce non-equilibrium behaviour. The explicit solution of the system $d = 1$ with rates (2.8a) revealed that there then exists a mapping onto an equivalent equilibrium situation with an 'effective temperature', say T_{eff}, given by

$$\tanh(2J/k_{\mathrm{B}}T_{\mathrm{eff}}) = p\tanh(2J/k_{\mathrm{B}}T_1) + (1-p)\tanh(2J/k_{\mathrm{B}}T_2). \tag{2.10}$$

That specific study (Garrido *et al* 1987), however, provided no evidence for a similar mapping in other interesting, closely related cases, e.g. for $d = 1$ when the choice is (2.8b) and for $d = 2$ independently of the choice for $f_r(\boldsymbol{s})$.

That example illustrates both the great variety of situations one may encounter in practice when looking for a convenient object $E(\boldsymbol{s})$ and the outstanding interest those situations may bear in relation to the general theory of non-equilibrium phenomena. It will also serve to illustrate some of the advantages of the present method; namely, we shall achieve here general conclusions about that system without needing the solve each specific version of the model.

It may be mentioned that our method can in principle be applied also to the so-called probabilistic (spin-flip) cellular automata (Lebowitz *et al* 1990) whose evolution proceeds according to an equation of type (1.3c) and have no configurational energy similar to (2.5) defined. For instance, the one-dimensional case ($\Omega \equiv Z$)

$$P(\boldsymbol{s}|\boldsymbol{s}'; \delta t) = \prod_{i \in \Omega} \text{constant}[1 + s_{r_i} w(s'_{r_i})] \tag{2.11}$$

where w is any function such that $1 \pm w \geqslant 0$. This is not an illustration of (1.4), however, and some of our considerations below may not hold in that case (see, however, one of the examples in section 6).

3. Existence of stationary distributions

This section collects some results concerning the existence of an object $E(\boldsymbol{s})$, i.e. of the stationary probability distribution (2.2), for the general system defined in section 1; the situations in which $E(\boldsymbol{s})$ represents the system EH will be investigated in the following sections. Notice that we only consider here the case of a finite system, i.e. a finite number N of lattice sites. Nevertheless, when the resulting $E(\boldsymbol{s})$ does indeed represent the system EH, the only case of interest to us, this introduces no restriction at all in our main results given the property (2.4), as discussed below.

For a finite system with any dynamics c, the limit

$$\lim_{t\to\infty} P(s;\, t) > 0 \qquad s \in S \tag{3.1}$$

exists independent of the initial distribution of probabilities, $P(0) \equiv \{P(s;\, 0)\}_{s\in S}$.

The proof relies on the following facts. (a) Given that c satisfies (1.2), it follows that (1.1) (equivalently (1.4)) admits a unique solution with properties (1.3). (b) For any homogeneous Markov chain relating configurations s and s', it may only be that either $P(s|s';\, t) = 0$ for all t or else $P(s|s';\, t) > 0$ for all t. (c) It may be proved that the present case is characterised by $P(s|s';\, t) > 0$ for all $t > 0$ and any $s, s' \in S$; i.e. the Markov chain of interest here is irreducible. (d) For any irreducible Markov chain,

$$\lim_{t\to\infty} P(s|s';\, t) \equiv \Pi_s \qquad \text{independent of } s' \tag{3.2}$$

exists, with either $\Pi_s > 0$ for all s or else $\Pi_s = 0$ for all s, the latter case being possible only when the chain is infinite. (e) Given that

$$P(s;\, t) = \sum_{s'\in S} P(s|s';\, t) P(s';\, 0) \tag{3.3}$$

one may conclude that it exists

$$\begin{aligned} \lim_{t\to\infty} P(s;\, t) &= \sum_{s'\in S} \lim_{t\to\infty} P(s|s';\, t) P(s';\, 0) \\ &= \sum_{s'\in S} \Pi_s P(s';\, 0) = \Pi_s > 0. \end{aligned} \tag{3.4}$$

Facts (a), (b), (d) and (e) are familiar from the general theory of Markov processes (Haken 1977, van Kampen 1981, Ligget 1985). Property (c) may be proved as follows: Assume the chain is reducible, i.e. there are at least two configurations s and s' such that $P(s|s';\, t_0) = 0$ for $t_0 > 0$. Then, (b) implies that $P(s|s';\, t) = 0$ for all $t > 0$. Now, write the Kolmogorov equation (1.1b) as

$$\mathrm{d}P(s|s';\, t)/\mathrm{d}t = \sum_{s''\neq s'} c(s''|s') P(s|s'';\, t) + c(s'|s') P(s|s';\, t). \tag{3.5}$$

Given that $P(s|s';\, t) = 0$ for all $t > 0$, it follows from (3.5) that $0 = \Sigma_{s''\neq s'}\, c(s''|s')P(s|s'';\, t)$ for all $t > 0$, and given that one has from (1.2) that $s'' = s'^r$ implies that $c(s''|s') > 0$, it results $P(s|s'';\, t) = 0$ for all $t > 0$ and any $s'' = s'^r$ with $r \in \Omega$. That is, assuming s cannot be reached from s', it follows that s cannot be reached from any configuration obtained from s' by flipping a single spin, say. Now, s may be reached from s' by flipping at most N spins, so that, by iterating the above reasoning, one has that $P(s|s;\, t) = 0$ for all $t > 0$ and, consequently, that $\mathrm{d}P(s|s;\, t)/\mathrm{d}t = 0$ for all $t > 0$. This implies that, against the hypothesis $c(s|s) = -\Sigma_{s'}\, c(s|s') < 0$, that there exists

$$\lim_{t\to 0} \mathrm{d}P(s|s;\, t)/\mathrm{d}t = 0 \equiv c(s|s). \tag{3.6}$$

As a corollary, it follows that a system with N sites and any dynamics c admits a unique object $E(s)$ given by (2.3). Actually, it suffices to take

$$J^{(k)}_{i_1 \dots i_k} = (2^N)^{-1} \sum_{s\in S} s_{i_1} \dots s_{i_k} \ln \Pi_s \tag{3.7}$$

to obtain $P^{\mathrm{st}}(s) = Z^{-1} \exp[-H(s)] = \Pi_s$.

That is, the above theorem not only asserts the existence of an object $E(s)$ (i.e. the existence of a stationary state) for each choice of the transition rates $c(s^r|s)$ but it also secures its uniqueness independently of the initial probability distribution, and that the system will approach stationarity as $t \to \infty$ for any specific choice for the matrix elements $c(s^r|s)$. Two problems still remain when $E(s)$ exists, however: (a) it may occur in general that $E(s)$ includes an indefinite number of terms as $N \to \infty$, and (b) when $E(s)$ is the EH, a method is needed to compute the coefficients $J^{(k)}$ in (2.3). Concerning the former problem, we have already noticed that our purposes here dictate that $E(s)$ needs to have a short-ranged nature as in (2.4); fortunately, this is indeed the case in many models of interest, as will be shown below. The latter problem requires in general to solve a system of (2^N) homogeneous linear equations, compute P^{st}, and use (3.7). To avoid such a lengthy and, sometimes, unrealisable procedure, we shall mainly be concerned (see, however, section 8) in cases where a global detailed balance (GDB) property holds. That is, unless otherwise indicated we shall assume that the functions defined in (1.5) and (2.2) satisfy that

$$c(s|s^r)P^{st}(s^r) = c(s^r|s)P^{st}(s) \qquad \text{for all } s \text{ and } s^r. \tag{3.8}$$

When (3.8) holds, one simply has that

$$\ln[c(s^r|s)/c(s|s^r)] = -\delta E = 2s_r \sum_{k=1}^{N} \sum_{(r,i_2\dots i_k)}' J^{(k)}_{r,i_2\dots i_k} s_{i_2} \dots s_{i_k} \tag{3.9}$$

and the unknowns $J^{(k)}$ follow by identifying coefficients. Consequently, the following result is interesting.

The necessary and sufficient condition for any dynamics c to fulfil the GDB condition (3.8) is that any set of k spin variables satisfies that

$$\sum_{s\in S} s_{i_1}s_{i_2}\dots s_{i_j}s_{i_m}\dots s_{i_k} \ln[c(s^{i_j}|s)c(s|s^{i_m})/c(s|s^{i_j})c(s^{i_m}|s)] = 0 \tag{3.10}$$

for all i_j and i_m. Such a dynamics c may then be written as

$$c(s^r|s) = f_r(s)\exp(-\tfrac{1}{2}\delta E) \qquad \delta E \equiv E(s^r) - E(s) \tag{3.11}$$

where $f_r(s) = f_r(s^r) > 0$, i.e. $f_r(s)$ has no dependence on the value of s_r.

That (3.10) is a necessary condition follows from the facts: (a) the coefficients $J^{(k)}$ remain unchanged when permuting their subindexes, and (b) given that (3.9) needs to hold, one has that

$$J^{(k)}_{i_1\dots i_j\dots i_k} = B^{(k,j)} \equiv (2^{N+1})^{-1} \sum_{s\in S} s_{i_1}\dots s_{i_k} \ln[c(s^{i_j}|s)/c(s|s^{i_j})] \tag{3.12}$$

where we have avoided the use of the indices $i_1 \dots i_j \dots i_k$ in the notation for $B^{(k,j)}$. The proof that (3.10) is sufficient follows by noticing that, for any set of k sites, one has that $B^{(k,1)} = B^{(k,2)} = \dots = B^{(k,k)} \equiv B^{(k)}$ as a consequence of (3.10). Thus, $B^{(k)}$ remains unchanged when permuting $(i_1, \dots, i_k)$ and, when one chooses

$$E(s) \equiv \sum_{k=1}^{N} \sum_{(i_1\dots i_k)}' B^{(k)} s_{i_1} \dots s_{i_k} \tag{3.13}$$

it follows that $\ln[c(s^{i_j}|s)/c(s|s^{i_j})] - [E(s) - E(s^{i_j})] = 0$ which implies that $E(s)$ satisfies (3.9).

4. Effective Hamiltonians

Once the existence of an explicit stationary probability distribution has been demonstrated under rather general conditions, we may investigate the cases for which the number of coefficients $J^{(k)}$ in $E(s)$ remains constant independent of N, the number of sites in the system. There is a large and interesting class of situations having that property when the system dynamics satisfies GDB. The following holds.

The object $E(s)$ is the EH of the system when the resulting transiton rate (1.5), say

$$c(s^r|s) = \sum_{l=1}^{m} c_l(s^r|s) \tag{4.1}$$

satisfies the condition (3.8) of GDB *and* depends only on a constant number of sites which is independent of N. Indeed, (3.8) implies that the coefficients $J^{(k)}$ are given by (3.12). Thus, when $c(s^r|s)$ and $c(s|s^r)$ only depend on M spin variables, only the coefficients $J^{(k)}$ involving those M variables will differ from zero, and there are at most 2^M coefficients having that property.

The latter theorem reveals the great usefulness of condition (3.8). That is, when GDB is satisfied, it is quite simple to investigate the existence of an EH. Consequently, our interest turns now to the analysis of that property. We already stated in (3.10) the necessary and sufficient condition for (3.8) to hold in the most general case. In order to be more explicit, and prepare our formalism to consider some specific examples, we study in detail in the next section the situations in which that holds in the case of an interesting and rather general one-dimensional system.

5. A generalised one-dimensional system

Consider the system defined in section 1 with $d=1$. The dynamics consists of a competition as in (4.1) with

$$c_l(s^i|s) = f_i^{(l)}(s)\exp\{-\tfrac{1}{2}[H_l(s^i)-H_l(s)]\}. \tag{5.1}$$

Here, the functions f_i are assumed to be analytic, positive defined and independent of the variable s_i, so that each individual transition rate $c_l(s^i|s)$ satisfies the condition of detailed balance (2.7). It is further assumed that the individual, actual physical Hamiltonians involve a term corresponding to the action of an external magnetic field, i.e.

$$H_l(s) = -K_1^{(l)}\sum_{i=1}^{N} s_i - K_2^{(l)}\sum_{i=1}^{N} s_i s_{i+1} \tag{5.2}$$

and that the functions f_i satisfy the following properties. (i) They are invariant under the interchange of s_{i-1} and s_{i+1}; consequently, they may be written as

$$f_i^{(l)}(s) = g_0^{(l)}(s_{il}) + g_1^{(l)}(s_{il})\sigma_1^i + g_2^{(l)}(s_{il})\sigma_2^i \tag{5.3}$$

where $\sigma_1^i \equiv \tfrac{1}{2}(s_{i-1}+s_{i+1})$, $\sigma_2^i \equiv s_{i-1}s_{i+1}$, and s_{il} represents the set of occupation variables in $f_i^{(l)}$ excluding s_{i-1} and s_{i+1}. (ii) They are homogeneous in the sense that the coefficients g in (5.3) are independent of i, as it is already reflected in our notation. (iii) They are symmetric in the sense that, given $m^l < \tfrac{1}{2}(N-1)$, $s_{i+m}^l \in s_{il}$ implies that $s_{i-m}^l \in s_{il}$, and $f_i^{(l)}$ has precisely the same dependence on both, s_{i+m}^l and s_{i-m}^l. (iv) Each individual dynamics has a few-body nature in the sense that it involves no 'many' neighbours of site i, i.e. $\max_{(l)}\{m^l \mid s^l_{i+m} \in s_{il}\} \equiv M \ll \frac{1}{2}(N-1)$ as it certainly occurs in every familiar case, e.g. for the rates by Glauber (1963), de Masi *et al* (1985), Kawasaki (1972), Metropolis *et al* (1953) and by van Beijeren and Schulman (1984).

Our interest in the definition in the last paragraph rests upon the fact that, as follows from the general results in the previous sections, the object $E(s)$ always exists for any one-dimensional model system with such a dynamics. Let us study now what conditions satisfy GDB so that $E(s)$ is the system EH.

One may show after some algebra that

$$\ln[c(s|s^i)/c(s^i|s)] = s_i[D_0(s_i^*) + 2D_1(s_i^*)\sigma_1^i + D_2(s_i^*)\sigma_2^i] \tag{5.4}$$

$$D_0(s_i^*) \equiv \tfrac{1}{4}(b+d+2a) \qquad D_1(s_i^*) \equiv \tfrac{1}{4}(b-d) \qquad D_2(s_i^*) \equiv \tfrac{1}{4}(b+d-2a) \tag{5.5}$$

$$a(s_i^*) \equiv \ln\{\Sigma_l[g_0^{(l)} - g_2^{(l)}]\exp[K_1^{(l)}]/\Sigma_l[g_0^{(l)} - g_2^{(l)}]\exp[-K_1^{(l)}]\} \tag{5.6a}$$

$$b(s_i^*) \equiv \ln\left(\frac{\Sigma_l[g_0^{(l)} + g_1^{(l)} + g_2^{(l)}]\exp[K_1^{(l)} + 2K_2^{(l)}]}{\Sigma_l[g_0^{(l)} + g_1^{(l)} + g_2^{(l)}]\exp[-K_1^{(l)} - 2K_2^{(l)}]}\right) \tag{5.6b}$$

$$d(s_i^*) \equiv \ln\left(\frac{\Sigma_l[g_0^{(l)} - g_1^{(l)} + g_2^{(l)}]\exp[K_1^{(l)} - 2K_2^{(l)}]}{\Sigma_l[g_0^{(l)} - g_1^{(l)} + g_2^{(l)}]\exp[-K_1^{(l)} + 2K_2^{(l)}]}\right) \tag{5.6c}$$

$l = 1, 2, \ldots, m$. Notice that s_i^* represents the set of spin variables appearing in a, b and d, and that s_i^* may differ from the set s_{il}; in any case, one still has the properties (i)–(iv) above. Thus, a necessary and sufficient condition in order to have GDB here is that, for all j and $k \neq j$:

$$\sum_{s} s_{i_1} \ldots s_{i_m} G^{j,k}(s) = 0 \qquad i_1 \neq j, k \qquad l = 1, \ldots, m \qquad m \geq 0 \tag{5.7}$$

where the functions $G^{j,k}(s)$, which are defined as

$$G^{j,k}(s) \equiv s_j s_k \ln[c(s^j|s)c(s|s^k)/c(s|s^j)c(s^k|s)] \tag{5.8}$$

are given for the model in this section as

$$G^{j,k}(s) = s_j[D_0(s_k^*) + 2D_1(s_k^*)\sigma_1^k + D_2(s_k^*)\sigma_2^k] - s_k[D_0(s_j^*) + 2D_1(s_j^*)\sigma_1^j + D_2(s_j^*)\sigma_2^j]. \tag{5.9}$$

Then, taking k as an NN of j, e.g. $k = j-1$, it results that the condition of GDB implies that

$$D_1(s_{j-1}^*) + D_2(s_{j-1}^*)s_{j-2} = D_1(s_j^*) + D_2(s_j^*)s_{j+1}. \tag{5.10}$$

Now, the only way to satisfy (5.9) is by requiring that

$$D_1(s_{j-1}^*) = D_1(s_j^*) = \text{constant} \tag{5.11a}$$

and

$$D_2(s_{j-1}^*) = D_2(s_j^*) = 0. \tag{5.11b}$$

The proof is as follows. Given the homogeneity (ii) of the functions $f_i^{(l)}$ and the definitions (5.5), one may write that

$$D_1(s_j^*) = \sum_{m \geq 0} \sum_{(i_1 \ldots i_m)}' B^m_{i_1 \ldots i_m} s_{j+i_1} \ldots s_{j+i_m} \tag{5.12a}$$

$$D_2(s_j^*) = \sum_{m \geq 0} \sum_{(i_1 \ldots i_m)}' C^m_{i_1 \ldots i_m} s_{j+i_1} \ldots s_{j+i_m} \tag{5.12b}$$

and it may be proved that $B^m = C^m = 0$ for all $m \geqslant 1$. Indeed, let us assume that, on the contrary, there exists some m, $i_1, \ldots, i_m$, for which B^m and C^m differ from each other or they are non-zero. The few-body nature (iv) of $f_i^{(l)}$ implies that there exists $\max_{m\geqslant 1}\{i_1, \ldots, i_m \,|\, B^m \neq C^m \text{ or } B^m = C^m \neq 0\} \equiv M^*$, and the symmetry property (iii) together with the fact that $B^m_{1,i_2\ldots i_m} = C^m_{1,i_2\ldots i_m} = 0$ for all $m \geqslant 1$ implies that $M^* > 1$. Thus, the RHS of (5.10) depends on s_{j+M^*} (also on s_{j-M^*} and s_{j+1}), while the LHS of (5.10) cannot depend on s_{j+M^*} given the homogeneity property (ii) so that one also has that $B^m_{i_1\ldots i_m} = C^m_{i_1\ldots i_m} = 0$ for all $m \geqslant 1$ and, consequently, that $D_1 = \text{constant}$ and $D_2 = 0$. QED.

Summing up, a necessary condition for the present model to satisfy GDB is that

$$\ln[c(s|s^i)/c(s^i|s)] = s_i[D_0(s_i^*) + 2D_1\sigma_1^i] \tag{5.13}$$

where D_1 is a constant and

$$D^0(s_j^*) = \sum_{m\geqslant 0} \sum_{(i_1\ldots i_m)}' A^m_{i_1\ldots i_m} s_{j+i_1} \ldots s_{j+i_m} \tag{5.14}$$

where one should notice that the subscripts $i_1 \ldots i_m$ refer to the distance to the j-spin, remains undetermined in principle. It is then rather simple to prove that, assuming also that the coefficients A^m satisfy

$$A^m_{i_1\ldots i_l\ldots i_m} = A^m_{i_1-i_l\ldots -i_l\ldots i_m+i_l} \tag{5.15}$$

the condition (5.13) becomes necessary and sufficient. Indeed, (5.9) reduces to

$$G^{j,k}(s) = 2D_1[s_j\sigma_1^k - s_k\sigma_1^j] + [s_jD_0(s_k^*) - s_kD_0(s_j^*)] \tag{5.16}$$

where the first bracket always satisfies, for all j, k and $m \geqslant 0$, that

$$\sum_{s} s_{i_1} \ldots s_{i_m}[s_j\sigma_1^k - s_k\sigma_1^j] = 0 \qquad i_l \neq j, k \tag{5.17}$$

and the second bracket has a similar property when $s_j \notin s_k^*$ or $s_k \notin s_j^*$. Consequently, GDB requires that, for any L such that $s_{j+L} \in s_j^*$, or equivalently that $s_j \in s_{j+L}^*$, one has the property

$$\sum_{s} s_{j+k_1} \ldots s_{j+k_n}[s_jD_0(s_{j+L}^*) - s_{j+L}d_0(s_j^*)] = 0 \tag{5.18}$$

for any $n \geqslant 0$ and any $k_l \neq 0, L$. Then, the condition (5.15) follows immediately when one writes $D_0(s_{j+L}^*)$ and $D_0(s_j^*)$ in terms of the coefficients A^m.

Notice that, in addition, to find a necessary and sufficient condition for GDB to hold in the generalised one-dimensional model considered in the present section, we concluded here about the expression for the corresponding EH. Indeed, that readily follows now from (3.12), (5.13) and (5.15); one has in particular that $J^{(2)}_{i,i\pm 1} = -\frac{1}{2}D_1$. It is further noticeable the fact that we also concluded $D_2 = 0$. This implies in particular that, for $d = 1$, any existing EH when GDB holds needs to have the familiar nearest-neighbour Ising structure. The latter was concluded before by Garrido and Marro (1989) for $d = 1$ and 2 under some restrictions for the functions f_i in (5.1).

6. Some specific examples for $d \geqslant 1$

Consider first the simplest one-dimensional case in which every rate (5.1) is defined with respect to the same 'Hamiltonian' (5.2), i.e. $H_1(s) \equiv H(s)$, $K_1^{(l)} \equiv h/k_BT$ and

$K_2^{(l)}K \equiv J/k_BT$. It follows that $\ln[c(s|s^i)/c(s^i|s)] = 2s_i[h/k_BT + 2K\sigma_1^i]$, so that GDB is (trivially) satisfied, $J_i^{(1)} = -h/k_BT$, $J_{i,i\pm1}^{(2)} = -K$, and the rest and coefficients are zero; that is, $E(s) = H(s)$. Notice also that the specific choice for the function $f_i^{(l)}(s)$ in (5.1) is irrelevant in this case.

More interesting is the situation in which the dynamics is a mixture (2.9) of two rates (5.1) each defined with respect to a different 'Hamiltonian', $H_l(s)$, $l = 1, 2$. Consider for instance the case solved before by Garrido *et al* (1987) which may be characterised by $K_1^{(l)} \equiv 0$, $K_2^{(l)} \equiv \varphi_l J$ with $\varphi_1 = [k_B(T - \delta T)]^{-1}$ and $\varphi_2 = [k_B(T + \delta T)]^{-1}$, and

$$c_l(s^i|s) = \{\cosh[2K_2^{(l)}\sigma_1^i]\}^{-1} \exp\{2K_2^{(l)}\sigma_1^i s_i\} \tag{6.1}$$

which corresponds to (2.8*a*) and $d = 1$. It follows rather straightforwardly from the results in the previous section that $D_0 = D_2 = 0$ and $D_1 = \frac{1}{2}b$, with

$$b = \ln\{[1 + p\alpha_1 + (1-p)\alpha_2]/[1 - p\alpha_1 - (1-p)\alpha_2]\} \qquad \alpha_l \equiv \tanh(2K\varphi_l) \tag{6.2}$$

and, consequently, that $J_{i,i\pm1}^{(2)} = -\frac{1}{4}b$ is the only non-zero coefficient in (2.3). Thus, by defining an 'effective temperature' T_{eff} such that $J_{i,i\pm1}^{(2)} = -J/K_BT_{\text{eff}}$, one has that $\tanh(2J/k_BT_{\text{eff}}) = p\alpha_1 + (1-p)\alpha_2$. This is precisely the result in (2.10) which was only obtained before (Garrido *et al* 1987) after explicitly solving the model. The relative simplicity of the present method also becomes evident by considering different choices for $f_i^{(l)}(s)$ in (5.1) other than (6.1). For instance, the choice (2.8*d*) ($d = 1$) immediately leads to

$$(T_{\text{eff}})^{-1} = (k_B/4J)\ln\left(\frac{px_1^-/y_1^- + (1-p)x_2^-/y_2^-}{px_1^-/y_1^+ + (1-p)x_2^-/y_2^+}\right) \tag{6.3}$$

where $x_l^\pm \equiv 1 + \exp(\pm 2J/k_BT_l)$ and $y_l^\pm \equiv 1 + \exp(\pm 4J/k_BT_l)$, and assuming that $f_i^{(1)}(s)$ is of type (2.8*a*) and that $f_i^{(2)}(s)$ is of type (2.8*d*) leads to the same qualitative situation except that

$$b = \ln\left(\frac{p(1+\alpha_1)\{1 + ([1-\alpha_1]/[1+\alpha_1])^{1/2}\} + 2(1-p)(1+\alpha_2)}{p(1+\alpha_1)\{1 + ([1-\alpha_1]/[1+\alpha_1])^{1/2}\} + 2(1-p)(1+\alpha_2)}\right) \tag{6.4}$$

where α_l was defined in (6.2). Interesting enough, we also find rather simply from the formulae in section 5 that

$$(T_{\text{eff}})^{-1} = (k_B/4J) \cdot \ln\left(\frac{p/(x_1^-)^2 + (1-p)/(x_2^-)^2}{p/(x_1^+)^2 + (1-p)/(x_2^+)^2}\right) \tag{6.5}$$

for the choice (2.8*b*), a one-dimensional case where the study by Garrido *et al* (1987) did not reveal the existence of any effective temperature.

A simple variation of the situation in the last paragraph may be characterised instead by $K_1^{(l)} \equiv 0$ and $K_2^{(l)} \equiv J_l/k_BT$, $J_1 = K$ and $J_2 = K + \delta K$. Clearly, one obtains the same formal results as before except that the need is now for an 'effective interaction' $J_{\text{eff}} = J_{i,i\pm1}^{(2)} = -\frac{1}{4}b$. In spite of that formal similarity, the present case bears a novel physical significance. This becomes evident when one considers the possibility of an external magnetic field h, i.e. $K_1^{(l)} \neq 0$. The model in the last paragraph would then require $K_1^{(1)} = h/(T - \delta T) \neq K_1^{(2)} = h/(T + \delta T)$, implying that GDB is not satisfied, while one has for the present model that $K_1^{(1)} = K_1^{(2)} = h/T$, so that GDB holds and there exists an EH whose only non-zero coefficients are $J_i^{(1)} = -h/k_BT$ and $J_{i,i\pm1}^{(2)}$, the latter being the same as for $h = 0$.

Next we consider the case of a mixture (2.9) of two dynamics (5.1) such that the associated 'Hamiltonians' are given by (5.2) with $K_1^{(1)}=h/k_BT$, $K_1^{(2)}=0$, and $K_2^{(1)}=K_2^{(2)}=K$, i.e. one acts with probability p as if the external magnetic field were h, and the other acts with probability $1-p$ as if there were no external field. It is interesting to refer, then, to different choices for the rates, i.e. for the functions $f_i^{(l)}(s)$ in (5.1), it readily follows that

$$D_0=\ln\left(\frac{p\exp(h/k_BT)+(1-p)}{p\exp(-h/k_BT)+(1-p)}\right)\qquad D_1=2K\qquad D_2=0.$$

Thus, GDB holds, there exists an EH whose only non-zero coefficients are $J_i^{(1)}=-\frac{1}{2}D_0$ and $J_{i,i\pm1}^{(2)}=-\frac{1}{2}D_1$, and one may define an 'effective field' $h_{\text{eff}}\equiv -J_i^{(1)}k_BT$. When the rate having an associated field is of type (2.8a) and the other rate is of type (2.8d), the situation is qualitatively similar. When both rates are given by (2.8c), the situation is, however, dramatically different, e.g. $D_2\neq 0$ implying that GDB is not satisfied. This illustrates, in particular, the outstanding role played by the details of the dynamics on the qualitative features of the (non-equilibrium) steady state.

As a further one-dimensional example we may consider the so-called voter model (see, for instance, Lebowitz and Saleur 1986) which belongs to the class of systems whose definition does not involve any Hamiltonian but a certain dynamical process. Namely, any configuration s evolves via spin-flips with a rate $c(s^i|s)$ such that, for $d=1$, it satisfies

$$\ln[c(s|s^i)/c(s^i|s)]=s_i\ln\left|\frac{1+(1-l)\sigma_1^i-l(1-2p)}{1-(1-l)\sigma_1^i+l(1-2p)}\right| \tag{6.6}$$

where $0\leqslant l,\ p\leqslant 1$. In the light of the results in section 5, this reveals in particular that GDB is only fulfilled either for $l=1$, when $E(s)=-\frac{1}{2}\ln[p/(1-p)]\Sigma_i s_i$, or else for $p=1/2$, when $E(s)=-\frac{1}{4}\ln[(2-l)/1]\Sigma_i s_i s_{i+1}$.

The general situation when $d=2$ may be analysed as discussed in sections 3 and 4, i.e. by following steps similar to the ones in section 5 and above for $d=1$. As an illustration, we refer here to the system driven by two heat baths at different temperatures in the case in which the rates are given by (2.8e) with

$$H_l(s)=-K_2^{(l)}\sum_{i=1}^{N}\sum_{j=1}^{M}(s_{ij}s_{i+1,j}+s_{ij}s_{i,j+1}) \tag{6.7}$$

where $K_2^{(l)}=\varphi_l J$. Our procedure readily uncovers that, excluding some trivial situations and the 'linear regime' considered in the next section, GDB is not satisfied by the system. This implies in particular that one may not define in general an effective temperature as in some cases before, in accordance with the indications in the study by Garrido *et al* (1987).

7. Small departures from equilibrium

Consider now a competing dynamics as in (4.1), or

$$c(s^r|s)=\sum_{i=1}^{m}p_ic_i(s^r|s)\qquad \sum_{i=1}^{m}p_i=1 \tag{7.1}$$

where each $c_i(s^r|s)$ satisfies a detailed balance condition (2.7) with respect to a different Hamiltonian $H(s;\boldsymbol{\alpha}_i)$. Here $\boldsymbol{\alpha}_i$ represents the set of parameters characterising a class of Hamiltonians; for instance, one may imagine this to be of the NN Ising type with an external field, (5.2), i.e. $\alpha_i = \{K_1^{(i)}, K_2^{(i)}\}$. Let us assume that each set of values $\boldsymbol{\alpha}_i$ differ by a small amount from some reference values, $\boldsymbol{\alpha}$,

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha} + \boldsymbol{\delta}_i \qquad \text{for every } i \tag{7.2}$$

so that one may expand the Hamiltonian as

$$H(\boldsymbol{s};\boldsymbol{\alpha}_i) = H(\boldsymbol{s};\boldsymbol{\alpha}) + (\mathrm{d}H/\mathrm{d}\boldsymbol{\alpha})\cdot\boldsymbol{\delta}_1 + \tfrac{1}{2}(\mathrm{d}^2H/\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\boldsymbol{\alpha}):\boldsymbol{\delta}_i\boldsymbol{\delta}_i + \ldots. \tag{7.3}$$

By using this expansion into condition (2.7), one has that

$$c_i(\boldsymbol{s}^r|\boldsymbol{s}) = c^0(\boldsymbol{s}^r|\boldsymbol{s}) + c_i^{(1)}(\boldsymbol{s}^r|\boldsymbol{s}) + c_i^{(2)}(\boldsymbol{s}^r|\boldsymbol{s}) + \ldots \tag{7.4}$$

where the transition rate contribution at each other, $c^0(\boldsymbol{s}^r|\boldsymbol{s})$ and $c_i^{(l)}(\boldsymbol{s}^r|\boldsymbol{s})$ with $l = 1, 2, \ldots$, satisfies a kind of detailed balance condition, namely that

$$c^0(\boldsymbol{s}^r|\boldsymbol{s})\exp\{-H(\boldsymbol{s};\boldsymbol{\alpha})\} = c^0(\boldsymbol{s}|\boldsymbol{s}^r)\exp\{-H(\boldsymbol{s}^r;\boldsymbol{\alpha})\} \tag{7.5}$$

$$\begin{aligned} c^0(\boldsymbol{s}^r|\boldsymbol{s})\exp\{-H(\boldsymbol{s};\boldsymbol{\alpha})\}[\Omega_i(\boldsymbol{s}) - \Omega_i(\boldsymbol{s}^r)] + c_i^{(1)}(\boldsymbol{s}^r|\boldsymbol{s})\exp\{-H(\boldsymbol{s};\boldsymbol{\alpha})\} \\ = c_i^{(1)}(\boldsymbol{s}|\boldsymbol{s}^r)\exp\{-H(\boldsymbol{s}^r;\boldsymbol{\alpha})\} \end{aligned} \tag{7.6}$$

$$\begin{aligned} c_i^{(2)}(\boldsymbol{s}^r|\boldsymbol{s})\exp\{-H(\boldsymbol{s};\boldsymbol{\alpha})\}[\Gamma_i(\boldsymbol{s}) - \Gamma_i(\boldsymbol{s}^r)\{\Omega_i(\boldsymbol{s}) - \Omega_i(\boldsymbol{s}^r)\}] \\ + c_i^{(1)}(\boldsymbol{s}^r|\boldsymbol{s})\exp\{-H(\boldsymbol{s};\boldsymbol{\alpha})\}[\Omega_i(\boldsymbol{s}) - \Omega_i(\boldsymbol{s}^r)] \\ + c_i^{(2)}(\boldsymbol{s}^r|\boldsymbol{s})\exp\{-H(\boldsymbol{s};\boldsymbol{\alpha})\} \\ = c_i^{(2)}(\boldsymbol{s}|\boldsymbol{s}^r)\exp\{-H(\boldsymbol{s}^r;\boldsymbol{\alpha})\} \end{aligned} \tag{7.7}$$

where

$$\Omega_i(\boldsymbol{s}) \equiv -[\mathrm{d}H(\boldsymbol{s};\boldsymbol{\alpha}_i)/\mathrm{d}\boldsymbol{\alpha}]\cdot\boldsymbol{\delta}_i \tag{7.8}$$

$$\Gamma_i(\boldsymbol{s}) \equiv \tfrac{1}{2}[\{\mathrm{d}H(\boldsymbol{s};\boldsymbol{\alpha}_i)/\mathrm{d}\boldsymbol{\alpha}\}\cdot\boldsymbol{\delta}_i]^2 - \tfrac{1}{2}[\mathrm{d}^2H(\boldsymbol{s};\boldsymbol{\alpha}_i)/\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\boldsymbol{\alpha}]:\boldsymbol{\delta}_i\boldsymbol{\delta}_i. \tag{7.9}$$

After using the result (7.4) in the expression (3.12) defining the effective Hamiltonian, one obtains that

$$E(\boldsymbol{s}) = H(\boldsymbol{s};\boldsymbol{\alpha}) + [\mathrm{d}H(\boldsymbol{s};\boldsymbol{\alpha})/\mathrm{d}\boldsymbol{\alpha}]\cdot[\Sigma_i p_i\boldsymbol{\delta}_i] + \mathrm{O}(\boldsymbol{\delta}^2). \tag{7.10}$$

That is, in a 'linear regime' where the parameters (such as interaction strength or external magnetic field) characterising the Hamiltonians associated to each transition rate via a detailed balance condition (2.7) have values close enough to a given set of values, it follows both (i) that GDB is satisfied by the system, and (ii) that there always exists an EH given by (7.10). Therefore, a system defined in such a linear regime, while being capable of a full non-equilibrium behaviour which may qualitatively differ from the equilibrium one, is expected to suffer 'small' departures from the canonical equilibrium associated with the reference Hamiltonian $H(\boldsymbol{s};\boldsymbol{\alpha})$ in the sense that it is content with most relevant canonical qualities, say (i) and (ii).

8. Discussion

We have studied interacting particle lattice systems whose configurations evolve in time according to a homogeneous Markov process, (1.1)-(1.4), which results from a competition between several Glauber, stochastic creation-annihilation mechanisms. Each particular Glauber mechanism acts as if it were associated with a different value of a given parameter, and it occurs at a rate given by (2.6)–(2.8). That is, each satisfies the detailed balance condition (2.7) which guarantees it would individually drive the system to the corresponding canonical equilibrium state. Nevertheless, that competition causes a more complex dynamics, e.g. the resulting transition rate (1.5) will not satisfy (2.7) in general, and the system may exhibit stationary non-equilibrium states as if it were acted on by some external agent.

Concerning finite lattices, we have shown that such a competing dynamics conserves probability and it allows the system when starting from any configuration to reach any other configuration in a finite number of steps. Consequently, there exists a unique, non-zero stationary probability distribution $P^{st}(s)$, one for each rate, and the system will tend asymptotically to it independently of the initial distribution. The function $P^{st}(s)$ may thus be written as in equations (2.2) and (2.3) defining $E(s)$, and it follows the uniqueness of $E(s)$. In the infinite-volume limit, $E(s)$ may not be unique. Nevertheless, we are only interested in cases in which $E(s)$ has a short ranged nature as in (2.4), i.e. in cases where the number and expressions of the coefficients $J^{(k)}$ in $E(s)$ are independent of the system size. While that short-ranged object $E(s)$ essentially differs from the actual Hamiltonian of the system under consideration, it represents the system 'effective Hamiltonian' (EH) in the sense of equations (2.2), e.g. it may be used to study non-equilibrium stationary states and phase transitions by applying standard methods of equilibrium statistical mechanics.

We have described a simple and systematic method to study the existence of an EH and to find explicit expressions for it. When the system evolving via a competition (1.5) of Glauber mechanisms satisfied the 'global detailed balance' (GDB) condition (3.8), a situation which is physically appealing and also very frequent in practice, there is always a unique short-ranged $E(s)$. We then find necessary and sufficient conditions for GDB to hold in the case of a one-dimensional ($d=1$) system, namely that $D_2=0$, $D_1=$ constant, and D_0 fulfils (5.15); cf equation (5.4). That is, the method readily allows one also to determine whether GDB holds or not when $d=1$, and there follows a similar way to verify that condition when $d>1$.

We have illustrated our method by studying several interesting one- and two-dimensional models. For $d=1$, we considered the so-called 'voter model' whose definition does not involve any configurational energy or Hamiltonian similar to (2.5) but a certain dynamical process. For $d=1$, we also considered three different modifications of the lattice-gas or Ising model with a competing dynamics as in (2.9). Namely, each $c_i(s^r|s)$, $i=1,2$, represents the rate for a Glauber change at site $\boldsymbol{r}$ performed assuming a given value for the bath temperature T, or for the particle (or spin) interaction strength J, or for the chemical potential (or applied magnetic field) h, respectively. The resulting physical situation is very rich and, interestingly enough, each of those four examples is such that GDB is only satisfied for some range of values of the system parameters or for certain families of transition rates $c_i(s^r|s)$. When that is the case, we obtain explicit expressions for $E(s)$, and it follows that the non-equilibrium system can be mapped on to an equivalent equilibrium one with some 'effective' value for the relevant parameter, say T, J or h. The case $d=2$ is illustrated by considering the system with two competing bath temperatures, an interesting situation which was studied before by more standard methods, namely Garrido *et al* (1987) reported the exact solution for $d=1$ and some approximate treatments for $d=2$. In particular, that study revealed no evidence for the existence of any equivalent equilibrium situation when the system is one-dimensional and $c_i(s^r|s)$ is given by (2.8*b*), nor when $d=2$ for any transition rate. Our method here, on the contrary, readily reveals the existence of an effective temperature, (6.5), in the former case and the fact that the two-dimensional system satisfies no GDB condition.

When GDB is not satisfied, it may still occur that there exists an EH and one may advise alternative methods, usually more indirect and specific ones, to determine the unique short-ranged object $E(s)$. For instance, one may relate in some cases the coefficients in $E(s)$ to some relevant correlation functions to be computed independently (see, for instance, Browne and Kleban 1989). Also, one may still follow the philosophy and formalism we developed before; e.g. when $D_2 \neq 0$ is small enough one may attempt an expansion around $D_2 = 0$. Such procedures, however, may strongly depend on the details of the system of interest; they usually involve lengthy algebraic manipulation, and they could not be enclosed so far in a general method, so that they are beyond the scope of this paper. In any case, we have also demonstrated the existence of a 'linear regime' where the non-equilibrium system may still retain most canonical features. We expect to report soon on further properties of the kinetically disordered systems considered here which have some relevance in relation with spin-glass, random-field and magnetically diluted models.

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